Part III Advanced Quantum Condensed Matter Physics

Question sheet IV

1. **Electron-phonon interaction in metals**
   Write a brief essay on the physical effects which can be produced by the electron phonon interaction in metals.

   - Electron-phonon scattering by phonon emission or absorption
   - Scattering time proportional to T at high T^{-1}, and proportional to T^{-3} at low temperature leading to temperature dependence of resistivity proportional to T at high T and T^{-5} at low T
   - Relaxation in transport processes, relaxation time approximation in Boltzmann equation
   - Effective attractive interaction between electrons mediated by phonons

2. **Canonical transformation of the electron-phonon coupling Hamiltonian**
   Show that for a canonical transformation of the Fröhlich Hamiltonian with transformation matrix

$$s = \sum_{\vec{k},\vec{k}'} \left(Aa^+_{-\vec{q}} + Ba_{\vec{q}}\right) M_{\vec{k}\vec{k}'} c^+_{\vec{k}'} c_{\vec{k}}$$

and coefficients $A = -\left(\varepsilon_{\vec{k}} - \varepsilon_{\vec{k}'} + \hbar\omega_{-\vec{q}}\right)^{-1}$ $\qquad B = -\left(\varepsilon_{\vec{k}} - \varepsilon_{\vec{k}'} - \hbar\omega_{\vec{q}}\right)^{-1}$

the following relationship between H_{el-ph}, H_0 and s is satisfied, which ensures that the transformed Hamiltonian $H' = e^{-S} H e^{S}$ only contains the electron-phonon-coupling in second order:

$$H_{el-ph} + \left[H_0, s\right] = 0$$

with $H_0 = \sum_{\vec{k}} \varepsilon\left(\vec{k}\right) c^+_{\vec{k}} c_{\vec{k}} + \sum_{\vec{q}} \hbar\omega(\vec{q}) a^+_{\vec{q}} a_{\vec{q}}$ $\qquad H_{el-ph} = \sum_{\vec{k},\vec{k}'} M_{\vec{k}\vec{k}'} \left(a^+_{-\vec{q}} + a_{\vec{q}}\right) c^+_{\vec{k}'} c_{\vec{k}}$

$$\left[\sum_{\vec{k}} \varepsilon\left(\vec{k}\right) c^+_{\vec{k}} c_{\vec{k}} + \sum_{\vec{q}} \hbar\omega(\vec{q}) a^+_{\vec{q}} a_{\vec{q}}, \sum_{\vec{k}'',\vec{k}'''} \left(Aa^+_{-\vec{q}'} + Ba_{\vec{q}'}\right) M_{\vec{k}''\vec{k}'''} c^+_{\vec{k}'''} c_{\vec{k}''}\right]$$

$$= \left[\sum_{\vec{k}} \varepsilon\left(\vec{k}\right) c^+_{\vec{k}} c_{\vec{k}}, \sum_{\vec{k}'',\vec{k}'''} \left(Aa^+_{-\vec{q}} + Ba_{\vec{q}}\right) M_{\vec{k}''\vec{k}'''} c^+_{\vec{k}'''} c_{\vec{k}''}\right] + \left[\sum_{\vec{q}} \hbar\omega(\vec{q}) a^+_{\vec{q}} a_{\vec{q}}, \sum_{\vec{k}'',\vec{k}'''} \left(Aa^+_{-\vec{q}'} + Ba_{\vec{q}'}\right) M_{\vec{k}''\vec{k}'''} c^+_{\vec{k}'''} c_{\vec{k}''}\right]$$

$$= \sum_{\vec{k}} \varepsilon\left(\vec{k}\right) \sum_{\vec{k}'',\vec{k}'''} \left(Aa^+_{-\vec{q}} + Ba_{\vec{q}}\right) M_{\vec{k}''\vec{k}'''} \left[c^+_{\vec{k}} c_{\vec{k}}, c^+_{\vec{k}'''} c_{\vec{k}''}\right] + \sum_{\vec{q}} \hbar\omega(\vec{q}) \sum_{\vec{k}'',\vec{k}'''} M_{\vec{k}''\vec{k}'''} c^+_{\vec{k}'''} c_{\vec{k}''} \left[a^+_{\vec{q}} a_{\vec{q}}, \left(Aa^+_{-\vec{q}'} + Ba_{\vec{q}'}\right)\right]$$

$$\left[a_q^+ a_q, A a_{-q'}^+ + B a_{q'}\right] = A a_q^+ a_q a_{-q'}^+ + B a_q^+ a_q a_{q'} - A a_{-q'}^+ a_q^+ a_q - B a_{q'} a_q^+ a_q$$
$$= A a_q^+ \left(\delta_{q-q'} + a_{-q'}^+ a_q\right) + B a_q^+ a_q a_{q'} - A a_{-q'}^+ a_q^+ a_q - B\left(\delta_{q'q} + a_q^+ a_{q'}\right) a_q$$
$$= A a_q^+ \delta_{q-q'} - B a_q \delta_{q'q}$$
$$\left[c_k^+ c_k, c_{k'''}^+ c_{k''}\right] = c_k^+ c_k c_{k'''}^+ c_{k''} - c_{k'''}^+ c_{k''} c_k^+ c_k$$
$$= c_k^+ \left(\delta_{k'''k} - c_{k'''}^+ c_k\right) c_{k''} - c_{k'''}^+ \left(\delta_{k''k} - c_k^+ c_{k''}\right) c_k$$
$$= c_k^+ c_{k''} \delta_{k'''k} - c_{k'''}^+ c_k \delta_{k''k}$$

$$\ldots = \sum_{\vec{k}} \varepsilon\left(\vec{k}\right) \sum_{\vec{k}'',\vec{k}'''} \left(A a_{-\vec{q}}^+ + B a_{\vec{q}}\right) M_{\vec{k}''\vec{k}'''} \left[c_{\vec{k}}^+ c_{\vec{k}}, c_{\vec{k}'''}^+ c_{\vec{k}''}\right] + \sum_{\vec{q}} \hbar\omega\left(\vec{q}\right) \sum_{\vec{k}'',\vec{k}'''} M_{\vec{k}''\vec{k}'''} c_{\vec{k}'''}^+ c_{\vec{k}''} \left[a_{\vec{q}}^+ a_{\vec{q}}, \left(A a_{-\vec{q}'}^+ + B a_{\vec{q}'}\right)\right]$$
$$= \sum_{\vec{k}} \varepsilon\left(\vec{k}\right) \sum_{\vec{k}'',\vec{k}'''} \left(A a_{-\vec{q}}^+ + B a_{\vec{q}}\right) M_{\vec{k}''\vec{k}'''} \left(c_k^+ c_{k''} \delta_{k'''k} - c_{k'''}^+ c_k \delta_{k''k}\right) + \sum_{\vec{q}} \hbar\omega\left(\vec{q}\right) \sum_{\vec{k}'',\vec{k}'''} M_{\vec{k}''\vec{k}'''} c_{\vec{k}'''}^+ c_{\vec{k}''} \left(A a_q^+ \delta_{q-q'} - B a_q \delta_{q'q}\right)$$
$$= \sum_{\vec{k}} \varepsilon\left(\vec{k}\right) \sum_{\vec{k}''} \left(A a_{-\vec{q}}^+ + B a_{\vec{q}}\right) M_{\vec{k}''\vec{k}} c_k^+ c_{k''} - \sum_{\vec{k}} \varepsilon\left(\vec{k}\right) \sum_{\vec{k}'''} \left(A a_{-\vec{q}}^+ + B a_{\vec{q}}\right) M_{\vec{k}\vec{k}'''} c_{k'''}^+ c_k +$$
$$+ \sum_{\vec{q}} \hbar\omega\left(\vec{q}\right) \sum_{\vec{k}'',\vec{k}'''} M_{\vec{k}''\vec{k}'''} c_{\vec{k}'''}^+ c_{\vec{k}''} A a_q^+ \delta_{q-q'} - \sum_{\vec{q}} \hbar\omega\left(\vec{q}\right) \sum_{\vec{k}'',\vec{k}'''} M_{\vec{k}''\vec{k}'''} c_{\vec{k}'''}^+ c_{\vec{k}''} B a_q \delta_{q'q}$$
$$= \sum_{\vec{k}k'} \varepsilon\left(\vec{k}\right) \left(A a_{-\vec{q}}^+ + B a_{\vec{q}}\right) M_{\vec{k}'\vec{k}} c_k^+ c_{k'} - \sum_{\vec{k}k'} \varepsilon\left(\vec{k}'\right) \left(A a_{-\vec{q}}^+ + B a_{\vec{q}}\right) M_{\vec{k}'\vec{k}} c_k^+ c_{k'}$$
$$+ \sum_{\vec{k}'',\vec{k}'''} \hbar\omega\left(-\vec{q}\right) M_{\vec{k}'\vec{k}} c_{\vec{k}}^+ c_{\vec{k}} A a_{-q}^+ - \sum_{\vec{q}} \sum_{\vec{k}'',\vec{k}'''} \hbar\omega\left(q\right) M_{\vec{k}'\vec{k}} c_{\vec{k}'}^+ c_{\vec{k}} B a_q$$
$$\Rightarrow A = -\left(\varepsilon_{\vec{k}} - \varepsilon_{\vec{k}'} + \hbar\omega_{-\vec{q}}\right)^{-1} \qquad B = -\left(\varepsilon_{\vec{k}} - \varepsilon_{\vec{k}'} - \hbar\omega_{\vec{q}}\right)^{-1}$$

3. AC conductivity of metals

Use the framework of Boltzmann theory to derive an expression for the AC conductivity of a homogeneous metal in response to an AC electrical field of the form: $\vec{E}\left(\vec{r},t\right) = \vec{E}_0 e^{i\vec{q}\cdot\vec{r}-\omega t}$

Hint: Look for solutions of the Boltzmann equation for the distribution function in the form $f_1\left(\vec{r},\vec{k},t\right) = \Phi\left(\vec{k}\right) e^{i\vec{q}\cdot\vec{r}-\omega t}$. *You can assume that the electrical field amplitude is sufficiently small to justify linearizing the Boltzmann equation.*

$$\vec{E}\left(\vec{r},t\right) = \vec{E}_0 e^{i\vec{q}\cdot\vec{r}-\omega t}: \qquad \vec{\nabla}_r f \cdot \vec{v} + \frac{1}{\hbar} \vec{\nabla}_k f \cdot \left(-e\vec{E}\right) + \frac{\partial f}{\partial t} = -\frac{f - f_0}{\tau}$$

Assume f = f_0 + f_1, where in lowest order f_1 is linear in the applied field. We can therefore replace f with f_0 in the ∇k – term:

$$\vec{\nabla}_r f_1 \cdot \vec{v} + \frac{1}{\hbar} \vec{\nabla}_k f_0 \cdot \left(-e\vec{E}\right) + \frac{\partial f_1}{\partial t} = -\frac{f_1}{\tau}$$

In an isotropic material f_1 can be assumed to have the same space and time dependence as the electric field:

$$f_1(\vec{r},\vec{k},t) = \Phi(\vec{k})e^{i\vec{q}\cdot\vec{r}-\omega t}: \qquad i\vec{q}\cdot\vec{v}\Phi(\vec{k}) + \frac{1}{\hbar}\vec{\nabla}_k f_0 \cdot (-e\vec{E}_0) - i\omega\Phi(\vec{k}) = -\frac{\Phi(\vec{k})}{\tau}$$

$$\Rightarrow \Phi(\vec{k}) = \frac{e\tau\vec{v}\cdot\vec{E}_0}{1-i\tau(\omega-\vec{q}\cdot\vec{v})}\frac{\partial f_0}{\partial E} \qquad \text{using } \vec{v} = \frac{1}{\hbar}\vec{\nabla}_k E(\vec{k})$$

$$\vec{J} = -\frac{e}{4\pi^3}\int d^3k\vec{v}f_1 = \frac{e^2}{4\pi^3}\int d^3k\tau(\vec{k})\left(-\frac{\partial f_0}{\partial E}\right)\vec{v}\frac{\vec{v}\cdot\vec{E}}{1-i\tau(\omega-\vec{q}\cdot\vec{v})} \equiv \vec{\sigma}\cdot\vec{E}$$

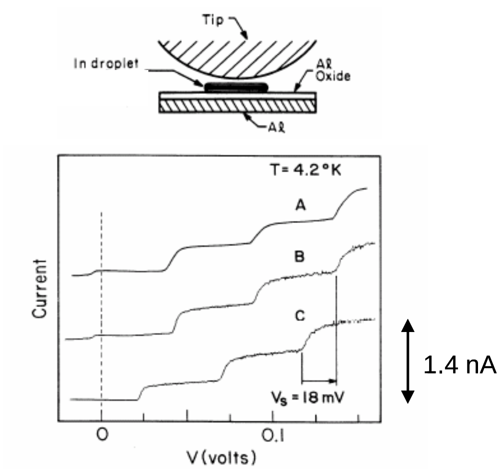
$$\vec{\sigma}(\vec{q},\omega) = \frac{e^2}{4\pi^3}\int d^3k\left(-\frac{\partial f_0}{\partial E}\right)\frac{\tau(\vec{k})\left(\vec{v}\cdot\vec{E}/|\vec{E}|\right)^2}{1-i\tau(\omega-\vec{q}\cdot\vec{v})}$$

- The conductivity tensor is determined only by electrons in a narrow shell near the Fermi level.
- In an anisotropic medium the current density and E do not need to be parallel, in cubic materials they are, however.
- In the semiclassical theory this expression is valid for both the longitudinal (E || q), as well as the transverse (E ⊥ q) conductivity function.

4. Classical and quantum transport

(a) Explain the conditions under which the wave-like nature of the electron can manifest itself in transport experiments.

(b) The figure below shows an experimental scanning tunneling spectrum of a double tunneling junction formed between the STM tip, an indium metallic island on an aluminum substrate with a 10 Å Al_2O_3 (ε=10) layer on it. From spectrum C (which is corrected with respect to the raw experimental spectrum A for offset voltages due to background charges) estimate the tunnel resistance and capacitance of the Al_2O_3 junction and the size of the In island by assuming that the resistance of the tunneling gap is smaller than that of the Al_2O_3 junction.



(a) Inelastic mean free path comparable to system size.

(b)

$$C_t = 2 \cdot 10^{-18} F = \frac{\varepsilon\varepsilon_0 \pi d^2}{4t} \quad \Rightarrow \quad d \approx 50A$$

$$R_t = \frac{e}{C_{tot}\Delta I} \approx \frac{e}{2C_t \Delta I} \approx 2 \cdot 10^8 \Omega$$

5. BCS theory of superconductivity

(a) Determine the energy eigenvalues and eigenvectors of the 2 x 2 matrix problem which determines the coefficients u_k and v_k of the BCS ground state wavefunction

$$\begin{pmatrix} \varepsilon_{\vec{k}} - \mu & \Delta_0 \\ \Delta_0^{\ *} & -(\varepsilon_{\vec{k}} - \mu) \end{pmatrix} \begin{pmatrix} u_{\vec{k}} \\ v_{\vec{k}} \end{pmatrix} = E_{\vec{k}} \begin{pmatrix} u_{\vec{k}} \\ v_{\vec{k}} \end{pmatrix}$$

(b) Show that the application of the quasiparticle operator $b_{k\sigma}$ applied to the BCS ground state is zero:

$$b_{\vec{k}\uparrow} \left| \Psi_{BCS} \right\rangle = 0$$

$$b_{\vec{k}\uparrow} \equiv u_{\vec{k}}^* c_{\vec{k}\uparrow} - v_{\vec{k}}^* c_{-\vec{k}\downarrow}^+ \qquad \left| \Psi_{BCS} \right\rangle = const \cdot \exp\left(\sum_{\vec{k}} \alpha_{\vec{k}} P_{\vec{k}}^+ \right) \left| 0 \right\rangle \qquad \text{with } P_{\vec{k}}^+ = c_{\vec{k}\uparrow}^+ c_{-\vec{k}\downarrow}$$

(a) Eigenvalues are determined by setting the determinant of M-E_kID equal to zero.

$$\det \begin{pmatrix} \varepsilon_{\vec{k}} - \mu - E_{\vec{k}} & \Delta_0 \\ \Delta_0^{\ *} & -(\varepsilon_{\vec{k}} - \mu) - E_{\vec{k}} \end{pmatrix} = 0$$

$$-\left(\varepsilon_{\vec{k}} - \mu - E_{\vec{k}}\right)\left(\varepsilon_{\vec{k}} - \mu + E_{\vec{k}}\right) - \left|\Delta_0\right|^2 = 0$$

$$E_{\vec{k}}^{\ 2} - \left(\varepsilon_{\vec{k}} - \mu\right)^2 - \left|\Delta_0\right|^2 = 0$$

$$E_{\vec{k}} = \sqrt{\left(\varepsilon_{\vec{k}} - \mu\right)^2 + \left|\Delta_0\right|^2}$$

$$-\left(\varepsilon_{\vec{k}}-\mu\right)u_{\vec{k}}-\Delta_0 v_{\vec{k}}+E_{\vec{k}}u_{\vec{k}}=0$$

$$\left(\varepsilon_{\vec{k}}-\mu\right)v_{\vec{k}}-\Delta_0^* u_{\vec{k}}+E_{\vec{k}}v_{\vec{k}}=0 \quad \Rightarrow u_{\vec{k}}=\frac{1}{\Delta_0^*}\left(\left(\varepsilon_{\vec{k}}-\mu\right)v_{\vec{k}}+E_{\vec{k}}v_{\vec{k}}\right)$$

$$u_{\vec{k}}/v_{\vec{k}}=\frac{1}{\Delta_0^*}\left(\left(\varepsilon_{\vec{k}}-\mu\right)+E_{\vec{k}}\right)$$

$$|u_{\vec{k}}|^2/|v_{\vec{k}}|^2=|u_{\vec{k}}|^2/\left(1-|u_{\vec{k}}|^2\right)=\frac{1}{|\Delta_0|^2}\left(\left(\varepsilon_{\vec{k}}-\mu\right)+E_{\vec{k}}\right)^2$$

$$|u_{\vec{k}}|^2=\frac{1}{|\Delta_0|^2}\left(\left(\varepsilon_{\vec{k}}-\mu\right)+E_{\vec{k}}\right)^2\left(1-|u_{\vec{k}}|^2\right)$$

$$|u_{\vec{k}}|^2\left(1+\frac{1}{|\Delta_0|^2}\left(\left(\varepsilon_{\vec{k}}-\mu\right)+E_{\vec{k}}\right)^2\right)=\frac{1}{|\Delta_0|^2}\left(\left(\varepsilon_{\vec{k}}-\mu\right)+E_{\vec{k}}\right)^2$$

$$|u_{\vec{k}}|^2=\frac{\left(\left(\varepsilon_{\vec{k}}-\mu\right)+E_{\vec{k}}\right)^2}{|\Delta_0|^2+\left(\left(\varepsilon_{\vec{k}}-\mu\right)+E_{\vec{k}}\right)^2}$$

$$|u_{\vec{k}}|^2=\frac{\left(\varepsilon_{\vec{k}}-\mu\right)^2+2\left(\varepsilon_{\vec{k}}-\mu\right)E_{\vec{k}}+E_{\vec{k}}^{\;2}}{E_{\vec{k}}^{\;2}-\left(\varepsilon_{\vec{k}}-\mu\right)^2+\left(\varepsilon_{\vec{k}}-\mu\right)^2+2\left(\varepsilon_{\vec{k}}-\mu\right)E_{\vec{k}}+E_{\vec{k}}^{\;2}}$$

$$|u_{\vec{k}}|^2=\frac{\left(\varepsilon_{\vec{k}}-\mu\right)^2+2\left(\varepsilon_{\vec{k}}-\mu\right)E_{\vec{k}}+E_{\vec{k}}^{\;2}}{2E_{\vec{k}}^{\;2}+2\left(\varepsilon_{\vec{k}}-\mu\right)E_{\vec{k}}}$$

$$|u_{\vec{k}}|^2=\frac{1}{2}\left(1+\frac{\left(\varepsilon_{\vec{k}}-\mu\right)^2+\left(\varepsilon_{\vec{k}}-\mu\right)E_{\vec{k}}}{E_{\vec{k}}^{\;2}+\left(\varepsilon_{\vec{k}}-\mu\right)E_{\vec{k}}}\right)$$

$$|u_{\vec{k}}|^2=\frac{1}{2}\left(1+\frac{\left(\varepsilon_{\vec{k}}-\mu\right)}{E_{\vec{k}}}\right)$$

$$|v_{\vec{k}}|^2=\frac{1}{2}\left(1-\frac{\left(\varepsilon_{\vec{k}}-\mu\right)}{E_{\vec{k}}}\right)$$

(b)

$$b_{\vec{k}\uparrow}\,|\Psi_{BCS}\rangle=\left(u_{\vec{k}}^*c_{\vec{k}\uparrow}-v_{\vec{k}}^*c_{-\vec{k}\downarrow}^+\right)|\Psi_{BCS}\rangle=\left(u_{\vec{k}}^*c_{\vec{k}\uparrow}-v_{\vec{k}}^*c_{-\vec{k}\downarrow}^+\right)\left(u_{\vec{k}}^*+v_{\vec{k}}^*c_{\vec{k}\uparrow}^+c_{-\vec{k}\downarrow}^+\right)|0\rangle$$

$$=\left(u_{\vec{k}}^*u_{\vec{k}}^*c_{\vec{k}\uparrow}+u_{\vec{k}}^*v_{\vec{k}}^*c_{\vec{k}\uparrow}c_{\vec{k}\uparrow}^+c_{-\vec{k}\downarrow}^+-u_{\vec{k}}^*v_{\vec{k}}^*c_{-\vec{k}\downarrow}^+-v_{\vec{k}}^*v_{\vec{k}}^*c_{-\vec{k}\downarrow}^+c_{\vec{k}\uparrow}^+c_{-\vec{k}\downarrow}^+\right)|0\rangle$$

$$=\left(u_{\vec{k}}^*u_{\vec{k}}^*c_{\vec{k}\uparrow}+u_{\vec{k}}^*v_{\vec{k}}^*\left(1-c_{\vec{k}\uparrow}^+c_{\vec{k}\uparrow}\right)c_{-\vec{k}\downarrow}^+-u_{\vec{k}}^*v_{\vec{k}}^*c_{-\vec{k}\downarrow}^+-v_{\vec{k}}^*c_{-\vec{k}\downarrow}^+c_{\vec{k}\uparrow}^+c_{-\vec{k}\downarrow}\right)|0\rangle=0$$

since none of the Bloch states is occupied in $|0\rangle$

6. Correlation length of superconductor

$$\langle \Psi_{BCS} | c^+_{\vec{k}\uparrow} c^+_{-\vec{k}\downarrow} | \Psi_{BCS} \rangle = \langle 0 | (u_{\vec{k}} + v_{\vec{k}} c_{-\vec{k}\downarrow} c_{\vec{k}\uparrow}) c^+_{\vec{k}\uparrow} c^+_{-\vec{k}\downarrow} (u^*_{\vec{k}} + v^*_{\vec{k}} c^+_{\vec{k}\uparrow} c^+_{-\vec{k}\downarrow}) | 0 \rangle$$
$$= \langle 0 | |u_{\vec{k}}|^2 c^+_{\vec{k}\uparrow} c^+_{-\vec{k}\downarrow} + u_{\vec{k}} v^*_{\vec{k}} c^+_{\vec{k}\uparrow} c^+_{-\vec{k}\downarrow} c^+_{\vec{k}\uparrow} c^+_{-\vec{k}\downarrow} + v_{\vec{k}} u^*_{\vec{k}} c_{-\vec{k}\downarrow} c_{\vec{k}\uparrow} c^+_{\vec{k}\uparrow} c^+_{-\vec{k}\downarrow} + |v_{\vec{k}}|^2 c_{-\vec{k}\downarrow} c_{\vec{k}\uparrow} c^+_{\vec{k}\uparrow} c^+_{-\vec{k}\downarrow} c^+_{\vec{k}\uparrow} c^+_{-\vec{k}\downarrow} | 0 \rangle$$
$$= v_{\vec{k}} u^*_{\vec{k}}$$

- Expectation value of the same operator in Fermi sea state would be zero because Fermi sea is a state of defined particle number [1].
- Energy range of order Δ_0; $\delta k = \dfrac{m\delta E}{\hbar^2 k_F}$ $\quad \xi = \dfrac{1}{\delta k} = \dfrac{\hbar^2 k_F}{m\Delta_0}$
- Number of Cooper pairs: $n_{cooper} \approx n \dfrac{\Delta_0}{E_F}$

$$k_F = 10^8 cm^{-1} \qquad \mathrm{n} = \frac{k_F^3}{3\pi^2} = 10^{23} cm^{-3}$$

- Typical numbers: $\Delta_0 = 1meV \quad \mathrm{E_F} = 1\mathrm{eV}$

  $\mathrm{n_{cooper}} = 10^{20} cm^{-3} \qquad \mathrm{n_{cooper}} \approx 1nm$

  $\xi = 1000nm$
- Many Cooper pairs within coherence length, the wavefunctions of individual Cooper pairs overlap