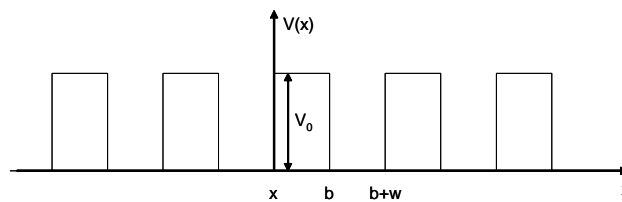


Part III Advanced Quantum Condensed Matter Physics

Question sheet I

1. Kronig-Penney model

Consider a one-dimensional, infinite periodic sequence of rectangular potential wells with lattice constant $a = w + b$.



Potential of the 1D Kronig-Penney model

(a) Show that the solutions of the Schrödinger equation for energies $0 < E < V_0$ have the form:

$$\psi_I(x) = Ae^{iqx} + Be^{-iqx} \quad -w < x < 0$$

$$\psi_{II}(x) = Ce^{\beta x} + De^{-\beta x} \quad 0 < x < b$$

Determine $q(E)$, $\beta(E)$ and derive equations to determine A, B, C, D by choosing the correct boundary conditions, including the condition for a Bloch electron of wavevector k . Explain how the dispersion relationship can be obtained from these equations.

$$q(E) = \sqrt{2mE/\hbar^2} \quad \beta(E) = \sqrt{2m(V_0 - E)/\hbar^2}$$

Boundary conditions :

$$\begin{aligned} \psi_I(0) = \psi_{II}(0) \quad \frac{d\psi_I}{dx} \Big|_{x=0} &= \frac{d\psi_{II}}{dx} \Big|_{x=0} \\ \psi_I(b) = e^{ika} \psi_I(-w) \quad \frac{d\psi_{II}}{dx} \Big|_{x=b} &= e^{ika} \frac{d\psi_I}{dx} \Big|_{x=-w} \quad (\text{Bloch theorem}) \end{aligned}$$

$$\Rightarrow A + B = C + D$$

$$Aiq - Biq = C\beta - D\beta$$

$$Ce^{\beta b} + De^{-\beta b} = e^{ika} [Ae^{-iqw} + Be^{iqw}]$$

$$C\beta e^{\beta b} - D\beta e^{-\beta b} = e^{ika} [Aiqe^{-iqw} - Biqe^{iqw}]$$

These equations have a solution only if the coefficient determinant vanishes :

$$\det \begin{pmatrix} 1 & 1 & -1 & -1 \\ iq & -iq & -\beta & \beta \\ -e^{ika-iqw} & -e^{ika+iqw} & e^{\beta b} & e^{-\beta b} \\ -iqe^{ika-iqw} & iqe^{ika+iqw} & \beta e^{\beta b} & -\beta e^{-\beta b} \end{pmatrix}$$

(b) By using the method described above it is possible to derive the following relationship between k and q (This requires some straightforward but somewhat tedious algebra which you are only requested to do if the weather is really miserable outside):

Life is too short to do this by hand (and the weather is nice too). I solved this in Mathematica:

$$m = \{ \{1, 1, -1, -1\}, \{Iq, -Iq, -\beta, \beta\}, \{-\text{Cos}[k^*a - q^*w] - I^* \text{Sin}[k^*a - q^*w], -\text{Cos}[k^*a + q^*w] - I^* \text{Sin}[k^*a + q^*w], e^{(\beta^*b)}, e^{(-\beta^*b)}\}, \{-I^*q^*(\text{Cos}[k^*a - q^*w] + I^* \text{Sin}[k^*a - q^*w]), I^*q^*(\text{Cos}[k^*a + q^*w] + I^* \text{Sin}[k^*a + q^*w]), \beta^*e^{(\beta^*b)}, -\beta^*e^{(-\beta^*b)}\} \};$$

$$\text{FullSimplify}[\text{Det}[m]]/4^*\beta^*e^{(\beta^*b)}$$

Result for determinant:

$$-\frac{1}{2} i \beta a e^{i a k} (-4 \beta a e^{b \beta} q \text{Cos}[a k] + 2 \beta a (1 + e^{2 b \beta}) q \text{Cos}[q w] + (-1 + e^{2 b \beta}) (\beta - q) (\beta + q) \text{Sin}[q w])$$

$$\frac{\beta^2 - q^2}{2q\beta} \sinh \beta b \sin qw + \cosh \beta b \cos qw = \cos ka$$

Simplify the above relationship by assuming a δ -like potential with $b \rightarrow 0$, and $V_0 b = \text{const}$, and show graphically in which energy bands allowed states can be found.

$$w = a \quad \beta \rightarrow \infty \quad \beta b \text{ is small}$$

$$\frac{\beta^2}{2q\beta} \beta b \sin qa + \cos qa = \cos ka$$

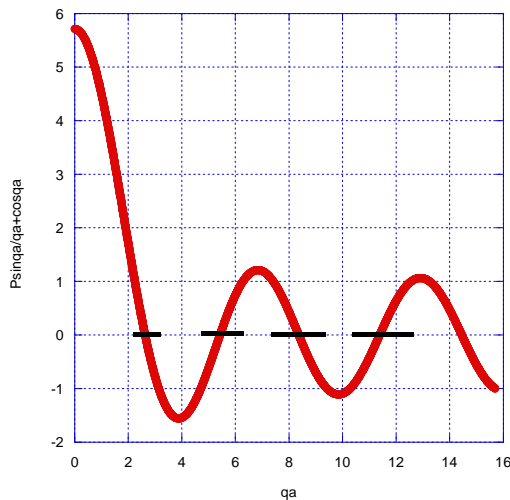
$$\frac{2mV_0 b}{2q\hbar^2} \sin qa + \cos qa = \cos ka$$

$$P \frac{\sin qa}{qa} + \cos qa = \cos ka \quad P = \frac{mV_0 b a}{\hbar^2}$$

(d) Plot the band structure for the four lowest energy bands for a value

$$V_0 = \frac{3\pi\hbar^2}{2mba}$$

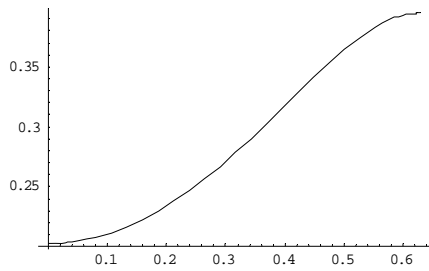
$$P = 3\pi/2$$



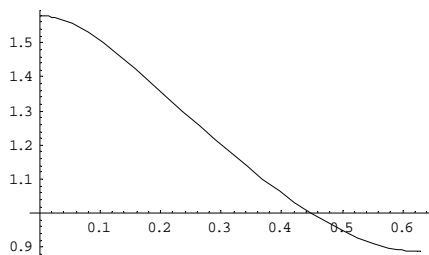
The black lines indicate bands in which Bloch solutions are possible, since $-1 < \cos(ka) < 1$. To get the band structure solve the above equation numerically (using Mathematica etc.) for a couple of values in the interval $k = -\pi/a$ to π/a . For each value of k a value of q is found, using $E = \hbar^2 q^2 / 2m$ the dispersion relation $E(k)$ is found.

```
h[k_]:=FindRoot[3*Pi/2*Sin[5*q]/5/q+Cos[5*q]==Cos[5k],{q,0.9,0.8,1.5},
WorkingPrecision->4,MaxIterations->100]
Plot[q^2/.h[k],{k,0,0.63}]
```

1st band (plotting q^2 vs k):



2nd band



etc.

(e) Consider the limiting case of very large and very small V_0 .

$P = 0$: free electron dispersion

$P \rightarrow \infty$: Allowed bands in graphical solution become very narrow. Energy spectrum composed of lines with:

$$q(E)a = n\pi \Rightarrow E_n = \frac{n^2 \hbar^2 \pi^2}{2ma^2} \quad \text{infinitely high quantum well with width } a$$

2. Boson creation and annihilation operators

Show that the creation and annihilation operators a and a^+ for the harmonic oscillator are boson operators and satisfy the appropriate commutation rules.

$$a = \sqrt{\frac{M\omega}{2\hbar}}u + i\sqrt{\frac{1}{2\hbar M\omega}}P$$

$$a^+ = \sqrt{\frac{M\omega}{2\hbar}}u - i\sqrt{\frac{1}{2\hbar M\omega}}P$$

$$\begin{aligned}
[a, a^+] &= aa^+ - a^+a = \left(\sqrt{\frac{M\omega}{2\hbar}}u + i\sqrt{\frac{1}{2\hbar M\omega}}P \right) \left(\sqrt{\frac{M\omega}{2\hbar}}u - i\sqrt{\frac{1}{2\hbar M\omega}}P \right) \\
&\quad - \left(\sqrt{\frac{M\omega}{2\hbar}}u - i\sqrt{\frac{1}{2\hbar M\omega}}P \right) \left(\sqrt{\frac{M\omega}{2\hbar}}u + i\sqrt{\frac{1}{2\hbar M\omega}}P \right) \\
&= \frac{M\omega}{2\hbar}uu - \frac{i}{2\hbar}uP + \frac{i}{2\hbar}Pu + \frac{1}{2\hbar M\omega}PP - \frac{M\omega}{2\hbar}uu - \frac{i}{2\hbar}uP + \frac{i}{2\hbar}Pu - \frac{1}{2\hbar M\omega}PP \\
&= \frac{i}{\hbar}[P, u] = 1
\end{aligned}$$

3. Independent electron approximation

Write a brief essay about solid state phenomena which are well described in the independent electron approximation, and discuss the conditions under which you would expect the independent electron approximation to break down.

- Band structure of many common, relatively wide-band s-/p-/d-orbital metals and semiconductors, and their optical and electrical properties.
- Nearly-free electron metals at high charge densities are well described in independent electron framework, because kinetic energy dominates, over exchange interaction. Breakdown at intermediate charge densities (spin polarization) and low charge densities, where electrons become localized in the field of the surrounding electrons (Wigner crystallization).
- In wide band systems kinetic energy tends to dominate. Breakdown of independent electron framework in narrow band systems, such as metals with partially filled f-shells, where interaction effects dominate over kinetic energy effects.
- Examples of phenomena that require going beyond the independent electron approximation: Hund's rule, collective phenomena such as magnetism, superconductivity; T^2 dependence of resistivity in heavy fermions, etc.

4. Slater determinant

(a) For a system of $N=3$ Hartree-Fock electrons in orbitals $\psi_i(\vec{r}, \sigma)$ ($i=1-3$) show by explicitly calculating the Slater determinant that the three-electron wavefunction is antisymmetric with respect to exchange of two electrons.

$$\begin{aligned}
\Psi &= \frac{1}{\sqrt{3!}} \begin{vmatrix} \psi_1(\vec{r}_1, \sigma_1) & \psi_1(\vec{r}_2, \sigma_2) & \psi_1(\vec{r}_3, \sigma_3) \\ \psi_2(\vec{r}_1, \sigma_1) & \psi_2(\vec{r}_2, \sigma_2) & \psi_2(\vec{r}_3, \sigma_3) \\ \psi_3(\vec{r}_1, \sigma_1) & \psi_3(\vec{r}_2, \sigma_2) & \psi_3(\vec{r}_3, \sigma_3) \end{vmatrix} \\
&= \frac{1}{\sqrt{6}} \begin{pmatrix} \psi_1(\vec{r}_1, \sigma_1)(\psi_2(\vec{r}_2, \sigma_2)\psi_3(\vec{r}_3, \sigma_3) - \psi_3(\vec{r}_2, \sigma_2)\psi_2(\vec{r}_3, \sigma_3)) \\ -\psi_1(\vec{r}_2, \sigma_2)(\psi_2(\vec{r}_1, \sigma_1)\psi_3(\vec{r}_3, \sigma_3) - \psi_3(\vec{r}_1, \sigma_1)\psi_2(\vec{r}_3, \sigma_3)) \\ +\psi_1(\vec{r}_3, \sigma_3)(\psi_2(\vec{r}_1, \sigma_1)\psi_3(\vec{r}_2, \sigma_2) - \psi_3(\vec{r}_1, \sigma_1)\psi_2(\vec{r}_2, \sigma_2)) \end{pmatrix} \\
\Psi((\vec{r}_1, \sigma_1) \leftrightarrow (\vec{r}_2, \sigma_2)) &= \frac{1}{\sqrt{6}} \begin{pmatrix} \psi_1(\vec{r}_2, \sigma_2)(\psi_2(\vec{r}_1, \sigma_1)\psi_3(\vec{r}_3, \sigma_3) - \psi_3(\vec{r}_1, \sigma_1)\psi_2(\vec{r}_3, \sigma_3)) \\ -\psi_1(\vec{r}_1, \sigma_1)(\psi_2(\vec{r}_2, \sigma_2)\psi_3(\vec{r}_3, \sigma_3) - \psi_3(\vec{r}_2, \sigma_2)\psi_2(\vec{r}_3, \sigma_3)) \\ +\psi_1(\vec{r}_3, \sigma_3)(\psi_2(\vec{r}_2, \sigma_2)\psi_3(\vec{r}_1, \sigma_1) - \psi_3(\vec{r}_2, \sigma_2)\psi_2(\vec{r}_1, \sigma_1)) \end{pmatrix} = -\Psi
\end{aligned}$$

(b) What is the quantum-mechanical interpretation of $\Psi^*\Psi$?

Probability of finding an electron at r_1 with spin σ_1 , a second electron at r_2 with spin σ_2 , and a third electron at r_3 with spin σ_3

(c) How can one calculate the total electron density at a particular location r for this many-electron wavefunction ? (What is requested here is the general formula you would use to calculate this, but not an explicit calculation.)

$$\begin{aligned}
n(\vec{r}) &= \langle \Psi(\vec{r}_1, \sigma_1, \vec{r}_2, \sigma_2, \vec{r}_3, \sigma_3) | \sum_i \delta(\vec{r} - \vec{r}_i) | \Psi(\vec{r}_1, \sigma_1, \vec{r}_2, \sigma_2, \vec{r}_3, \sigma_3) \rangle \\
&= \sum_{\sigma_1, \sigma_2, \sigma_3} \begin{pmatrix} \iint d^3r_2 d^3r_3 |\Psi(\vec{r}, \sigma_1, \vec{r}_2, \sigma_2, \vec{r}_3, \sigma_3)|^2 \\ + \iint d^3r_1 d^3r_3 |\Psi(\vec{r}_1, \sigma_1, \vec{r}, \sigma_2, \vec{r}_3, \sigma_3)|^2 \\ + \iint d^3r_1 d^3r_2 |\Psi(\vec{r}_1, \sigma_1, \vec{r}_2, \sigma_2, \vec{r}, \sigma_3)|^2 \end{pmatrix}
\end{aligned}$$

5. Derivation of Hartree-Fock equations

(a) Show that the expectation value for the energy of HF wavefunction comprising a Slater determinant of N -single electron orbitals $\{\psi_i(r, \sigma)\}$ is given by:

$$\begin{aligned}
E = \langle \Psi_0 | H | \Psi_0 \rangle &= \sum_{i=1}^N \left(\int d^3r \psi_i^*(\vec{r}, \sigma) \left(-\frac{\hbar^2}{2m} \Delta + V_{ion}(\vec{r}_i) \right) \psi_i(\vec{r}, \sigma) \right) \\
&+ \frac{1}{2} \sum_{i,j} \iint d^3r d^3r' |\psi_i(\vec{r}, \sigma)|^2 |\psi_j(\vec{r}', \sigma')|^2 \frac{e^2}{|\vec{r} - \vec{r}'|} \\
&- \frac{1}{2} \sum_{i,j} \iint d^3r d^3r' \psi_i^*(\vec{r}, \sigma) \psi_j^*(\vec{r}', \sigma') \frac{e^2}{|\vec{r} - \vec{r}'|} \psi_j(\vec{r}, \sigma) \psi_i(\vec{r}', \sigma') \delta_{\sigma_i, \sigma_j}
\end{aligned}$$

$$\Psi = \frac{1}{\sqrt{N!}} \begin{vmatrix} \psi_1(\vec{r}_1, \sigma_1) & \psi_1(\vec{r}_2, \sigma_2) & \cdots & \psi_1(\vec{r}_N, \sigma_N) \\ \psi_2(\vec{r}_1, \sigma_1) & \psi_2(\vec{r}_2, \sigma_2) & \cdots & \psi_2(\vec{r}_N, \sigma_N) \\ \cdots & \cdots & \cdots & \cdots \\ \psi_N(\vec{r}_1, \sigma_1) & \psi_N(\vec{r}_2, \sigma_2) & \cdots & \psi_N(\vec{r}_N, \sigma_N) \end{vmatrix}$$

$$= \frac{1}{\sqrt{N!}} \sum_{i=1}^{N!} (-1)^{\text{sign}(P_i)} P_i \{ \psi_1(\vec{r}_1, \sigma_1), \psi_2(\vec{r}_2, \sigma_2) \dots \psi_N(\vec{r}_N, \sigma_N) \}$$

The set of wavefunctions $\{\psi_1, \psi_2, \dots, \psi_N\}$ is orthonormal. For the one-electron kinetic energy & periodic potential energy operators only the term without a permutation from the fundamental order contributes. For the two-electron Coulomb energy operator there are two contributions. The first one arises by combining terms in Ψ and Ψ^* that have experienced the same permutation $i \leftrightarrow j$. This term is positive, since the two corresponding terms in Ψ and Ψ^* have the same sign. This gives the Hartree-term. Combining the term in Ψ with the fundamental order with the permutation in Ψ^* $i \leftrightarrow j$ gives the exchange term, but only if the spin of the two electrons is equal, since otherwise the orthogonality of the two spinors would render the integral zero. This term is negative, since the corresponding terms in Ψ and Ψ^* have different signs. All other terms vanish since at least one of the integrals $\langle \psi_k | \psi_l \rangle$ would be zero.

(b) Derive the Hartree-Fock equations by expressing the normalization conditions for the single-electron orbitals with a Lagrange multiplier ε_i and by using $\delta\psi_i$ and $\delta\psi_i^*$ as independent variable in a variational procedure.

$$E = \langle \Psi_0 | H | \Psi_0 \rangle = \sum_{i=1}^N \left(\int d^3r \psi_i^*(\vec{r}, \sigma) \left(-\frac{\hbar^2}{2m} \Delta + V_{ion}(\vec{r}_i) \right) \psi_i(\vec{r}, \sigma) \right. \\ \left. + \frac{1}{2} \sum_{i,j} \left(\iint d^3r d^3r' |\psi_i(\vec{r}, \sigma)|^2 |\psi_j(\vec{r}', \sigma')|^2 \frac{e^2}{|\vec{r} - \vec{r}'|} - \iint d^3r d^3r' \psi_i^*(\vec{r}, \sigma) \psi_j^*(\vec{r}', \sigma') \frac{e^2}{|\vec{r} - \vec{r}'|} \psi_j(\vec{r}, \sigma) \psi_i(\vec{r}', \sigma') \right) \delta_{\sigma_i, \sigma_j} \right)$$

Orthonormality condition : $\langle \psi_i | \psi_j \rangle = \delta_{ij}$

$$\delta E = \delta \left(\langle \Psi_0 | H | \Psi_0 \rangle - \sum_i \varepsilon_i (\langle \psi_i | \psi_i \rangle - 1) \right)$$

Taking $\delta\psi_i$ and $\delta\psi_i^*$ as independent variables so that $\delta\psi_i$ can be set to zero :

$$\langle \delta\psi_i | \left(-\frac{\hbar^2}{2m} \Delta + V_{ion}(\vec{r}_i) \right) | \psi_i \rangle + 2 \frac{1}{2} \sum_{j \neq i} \left[\langle \delta\psi_i | \psi_j | \frac{e^2}{r_{12}} | \psi_i \psi_j \rangle - \langle \delta\psi_i | \psi_j | \frac{e^2}{r_{12}} | \psi_j \psi_i \rangle \delta_{\sigma_i, \sigma_j} \right] - \sum_i \varepsilon_i \langle \delta\psi_i | \psi_i \rangle = 0$$

$$\left(-\frac{\hbar^2}{2m} \Delta + V_{ion}(\vec{r}) \right) \psi_i(\vec{r}) + \sum_{k \neq i} \int d^3r' |\psi_k(\vec{r}')|^2 \frac{e^2}{|\vec{r} - \vec{r}'|} \psi_i(\vec{r}) - \sum_{k \neq i} \int d^3r' \psi_k^*(\vec{r}') \psi_i(\vec{r}') \frac{e^2}{|\vec{r} - \vec{r}'|} \psi_k(\vec{r}) \cdot \delta_{\sigma_k, \sigma_i} = \varepsilon_i \cdot \psi_i(\vec{r})$$

6. Hartree-Fock ground state energy

Calculate the Hartree-Fock ground state energy for a set of occupied, single-particle Hartree-Fock orbitals and compare this value to the sum of their eigenvalues ε_i in the corresponding Hartree-Fock equations.

$$\begin{aligned}
E &= \langle \Psi | H | \Psi \rangle = \sum_{i=1}^N \left(\int d^3r \psi_i^*(\vec{r}) \left(-\frac{\hbar^2}{2m} \Delta + V_{ion}(\vec{r}) \right) \psi_i(\vec{r}) \right. \\
&\quad \left. + \frac{1}{2} \sum_{i,j} \left(\iint d^3r d^3r' |\psi_i(\vec{r})|^2 |\psi_j(\vec{r}')|^2 \frac{e^2}{4\pi\epsilon_0 |\vec{r} - \vec{r}'|} - \iint d^3r d^3r' \psi_i^*(\vec{r}) \psi_j^*(\vec{r}') \frac{e^2}{4\pi\epsilon_0 |\vec{r} - \vec{r}'|} \psi_j(\vec{r}) \psi_i(\vec{r}') \right) \delta_{\sigma_i, \sigma_j} \right) \\
&\left(-\frac{\hbar^2}{2m} \Delta + V_{ion}(\vec{r}) \right) \psi_i(\vec{r}) + \sum_{j \neq i} \int d^3r' |\psi_j(\vec{r}')|^2 \frac{e^2}{4\pi\epsilon_0 |\vec{r} - \vec{r}'|} \psi_i(\vec{r}) \\
&\quad - \sum_{j \neq i} \int d^3r' \psi_j^*(\vec{r}') \psi_i(\vec{r}') \frac{e^2}{4\pi\epsilon_0 |\vec{r} - \vec{r}'|} \psi_j(\vec{r}) \cdot \delta_{\sigma_j, \sigma_i} = \varepsilon_i \cdot \psi_i(\vec{r}) \\
E &= \sum_{i=1}^N \varepsilon_i - \frac{1}{2} \sum_{i,j} \left(\iint d^3r d^3r' |\psi_i(\vec{r})|^2 |\psi_j(\vec{r}')|^2 \frac{e^2}{4\pi\epsilon_0 |\vec{r} - \vec{r}'|} - \iint d^3r d^3r' \psi_i^*(\vec{r}) \psi_j^*(\vec{r}') \frac{e^2}{4\pi\epsilon_0 |\vec{r} - \vec{r}'|} \psi_j(\vec{r}) \psi_i(\vec{r}') \right) \delta_{\sigma_i, \sigma_j} \\
&\neq \sum_{i=1}^N \varepsilon_i
\end{aligned}$$

7. Proof of Koopman's theorem

Calculate the energy difference between the Hartree-Fock (HF) ground state of an N -electron system, and the HF ground state of an $N-1$ electron system obtained by removing an electron from one of the occupied HF orbitals of the N -electron system, and assuming that the remaining orbitals remain unchanged.

Take single HF orbital ψ_m out of the Slater determinant:

$$\begin{aligned}
E_0^{HF}(N) - E_0^{HF}(N-1) &= \langle \psi_m | h | \psi_m \rangle + \sum_{j, \text{occ}} \left[\langle \psi_m \psi_j | \frac{e^2}{4\pi\epsilon_0 r_{12}} | \psi_m \psi_j \rangle - \langle \psi_m \psi_j | \frac{e^2}{4\pi\epsilon_0 r_{12}} | \psi_j \psi_m \rangle \right] \\
&= \varepsilon_m
\end{aligned}$$

8. Hartree-Fock contact potential interaction

HF: Contact potential interaction

$$\Psi(\mathbf{r}) = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}_i} e^{i\mathbf{k}_i \cdot \mathbf{r}} \chi_{s_i}(\mathbf{r})$$

$$E = \sum_{\substack{\mathbf{k} < k_F \\ s_z = \frac{1}{2}}} \frac{\hbar^2 k^2}{2m_e} + \sum_{\substack{\mathbf{k} < k_F \\ s_z = -\frac{1}{2}}} \frac{\hbar^2 k^2}{2m_e} + \frac{1}{2} \sum_{\mathbf{r}} \int d^3r' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \Psi^\dagger(\mathbf{r}) \Psi(\mathbf{r}') - \frac{1}{2} \sum_{\mathbf{r}} \int d^3r' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \Psi^\dagger(\mathbf{r}) \Psi(\mathbf{r}') \delta_{\sigma, \sigma'}$$

$$= N \frac{3}{4} \frac{\hbar^2 k_F^2}{2m_e} + N \frac{3}{4} \frac{\hbar^2 k_F^2}{2m_e} + \frac{1}{2} \sum_{\substack{\mathbf{k} < k_F \\ s_z}} \frac{N \hbar^2}{V} - \frac{1}{2} \sum_{\substack{\mathbf{k} < k_F \\ s_z}} \frac{N s_z \hbar^2}{V}$$

* $\frac{4}{3} \pi k_F^3 = \frac{V}{(2\pi)^3} = N \Rightarrow k_F^3 = \left(\frac{6\pi^2 N}{V} \right)^{1/3}$
 $k_F^3 = \left(\frac{6\pi^2 N}{V} \right)^{1/3}$

$$\Rightarrow E = \frac{3}{5} \frac{\hbar^2}{2m_e} \left(\frac{6\pi^2}{V} \right)^{2/3} (N_\uparrow^{5/3} + N_\downarrow^{5/3}) + \frac{1}{2} \sum_{\substack{\mathbf{k} < k_F \\ s_z}} \frac{N s_z \hbar^2}{V}$$

$$= \frac{3}{5} \frac{\hbar^2}{2m_e} \left(\frac{6\pi^2}{V} \right)^{2/3} (N_\uparrow^{5/3} + N_\downarrow^{5/3}) + g \frac{N_\uparrow N_\downarrow}{V}$$

$$E_{\text{para}} = \frac{3}{5} \frac{\hbar^2}{2m_e} \left(\frac{6\pi^2}{V} \right)^{2/3} N^{5/3} + g \frac{N^2}{4V}$$

$$E_{\text{ferro}} = \frac{3}{5} \frac{\hbar^2}{2m_e} \left(\frac{6\pi^2}{V} \right)^{2/3} N^{5/3}$$

$$\frac{3}{5} \frac{\hbar^2}{2m_e} \left(\frac{6\pi^2}{V} \right)^{2/3} N^{5/3} \ll \frac{3}{5} \frac{\hbar^2}{2m_e} \left(\frac{6\pi^2}{V} \right)^{2/3} N^{5/3} + g \frac{N^2}{4V}$$

$$\left[\frac{2^{2/3}}{3} - 1 \right] \frac{3}{5} \left(\frac{\hbar^2}{2m_e} \right) \left(\frac{3\pi^2}{V} \right)^{2/3} N^{5/3} - g \frac{N^2}{4V} < 0$$

$$\left[\frac{2^{2/3}}{3} - 1 \right] \frac{3}{5} \left(\frac{\hbar^2}{2m_e} \right) \left(\frac{3\pi^2}{V} \right)^{2/3} - g \frac{N^{1/3}}{4V} < 0$$

$$N > \left[\frac{4}{9} (2^{2/3} - 1) \frac{3}{5} \left(\frac{\hbar^2}{2m_e} \right) \left(\frac{3\pi^2}{V} \right)^{2/3} \right]^3$$